A Note on Values for Markovian Coalition Processes

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Abstract

The Shapley value is defined as the average marginal contribution of a player, taken over all possible ways to form the grand coalition N when one starts from the empty coalition and adds players one by one. The authors have proposed in a previous paper an allocation scheme for a general model of coalition formation where the evolution of the coalition of active players is ruled by a Markov chain, and need not finish at the grand coalition. The aim of this note is to develop some explanations in the general context of time discrete stochastic processes, exhibit new properties of the model, correct some inaccuracies in the original paper, and give a new version of the axiomatization.

Keywords: coalitional game; coalition formation process; Shapley value
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1 Introduction

The *Shapley value* is a well-known allocation scheme for both TU- and NTU-games with numerous applications. It is defined as the average marginal contribution of a player, taken over all possible ways to form the grand coalition N when one starts from the empty coalition and adds players one by one.

In real situations however, there is no *a priori* reason for a process of cooperation to end with the grand coalition, nor are all ways of forming the grand coalition necessarily feasible. This explains why the Shapley value can produce counterintuitive results in some cases, as pointed out by, *e.g.*, Roth (1980), Shafer (1980), and Scafuri and Yannelis (1984).

Guided by these considerations, the authors have proposed an allocation scheme for a general model of coalition formation (Faigle and Grabisch, 2012) where the evolution of the coalition of active players is ruled by a Markov chain. The classical Shapley value appears then as the particular case where the only transitions possible consist

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of the addition of a single player to the present coalition and all these transitions are equiprobable.

The aim of this note is to develop some explanations in the even more general context of time discrete stochastic processes that are not necessarily Markovian, exhibit new properties of the model and correct some inaccuracies in the original paper (Faigle and Grabisch, 2012). In particular, we give a new version of the axiomatization. We restrict our exposition to the minimum, and refer the reader to the original paper for examples and further details on the Markovian model.

2 Coalition processes and values

We consider a finite set of players N, with $|N| = n$. By a *scenario* $\mathcal{S} = S_0, S_1, S_2, \ldots$ we mean a sequence of coalitions $S_t \subseteq N$ starting with the empty set $S_0 = \emptyset$. No particular property is assumed on the sequence (there could be repetitions for example). In this note, however, we will restrict ourselves to scenarios of finite length. We call any 2-element subsequence S_t, S_{t+1} in $\mathcal{S}$ a *transition* in $\mathcal{S}$ and denote it by $S_t \to S_{t+1}$.

A scenario $\mathcal{S}$ arises from the observation of the status of cooperation along (discrete) time $t = 0, 1, 2, \ldots$. We assume that a process of cooperation among players in N starts formally from the empty coalition $S_0 = \emptyset$ (no player is active), then coalition S_1 is observed, then S_2, *etc.* Coalition S_t is the set of active players (those engaged in cooperation or ready to cooperate) at time t. A finite scenario $\mathcal{S} = S_0, S_1, \ldots, S_\tau$ is said to be of *length* τ with S_τ being the final state of cooperation. Note that we do not necessarily assume $S_\tau = N$.

Example 1. Letting $N = \{1, 2, 3, 4\}$, consider the scenario

$$\mathcal{S} = \emptyset, 1, 14, 1, 123, 34$$

with the convention that 123 denotes $\{1, 2, 3\}$ *etc.* At time $t = 1$, player 1 becomes active and *enters* the current coalition. Then player 4 enters and is active at time $t = 2$ but becomes inactive at time $t = 3$ and *leaves* the current coalition. Next, the players 2 and 3 enter at the same time, while in the last time step $\tau = 4$, 1 and 2 leave and 4 enters. So, players 3 and 4 finally cooperate while the other players abstain from the game.

The example illustrates how our model captures the original idea of Shapley and generalizes it (see Faigle and Grabisch (2012) for real examples from exchange economies of Hart and Kurz (1983) and Scafuri and Yannelis (1984)).

We assume that scenarios are produced by some stochastic process, ruling the possible transitions between coalitions. In Faigle and Grabisch (2012), we have considered a Markov chain defined by a $2^n \times 2^n$ transition matrix $\mathbf{U} := [u_{S,T}]_{S,T \subseteq N}$, where $u_{S,T}$ is the probability of the transition $S \to T$ to occur if S is the currently active coalition. Therefore, the probability of a scenario $\mathcal{S} = \emptyset, S_1, \ldots, S_\tau$ to occur is simply

$$\Pr(\mathcal{S}) = \prod_{k=1}^{\tau} u_{S_{i-1}, S_i},$$

with $S_0 = \emptyset$. In general, we have probability distributions p^t on 2^N with $p^t(S)$ being the probability that S is the active coalition at time t. p^t can be viewed as the *state* of the coalition formation process at time t. Convergence to a limit state can be obtained by standard results in Markov chain theory.

We define an allocation scheme (value) for a general cooperation formation framework as follows. First, for any given scenario $\mathcal{S} = \emptyset, S_1, \dots, S_\tau$, we define what we call a *scenario-value*, that is, an allocation scheme for the considered scenario: $\psi^{\mathcal{S}} : \mathcal{G}(N) \to \mathbb{R}^n$, where $\mathcal{G}(N)$ is the set of TU-games on N. Then, the *value* ψ is the family $(\psi^{(\tau)}$ of the expectations of the scenario-values at time τ:

$$\psi^{(\tau)}(v) = \sum_{\mathcal{S}\in\mathfrak{S}_\tau} \Pr(\mathcal{S})\psi^{\mathcal{S}}(v),$$

where $\mathfrak{S}_\tau$ denotes the collection of all scenarios of length τ. Therefore, it suffices to concentrate on the definition of a suitable scenario-value. In (Faigle and Grabisch, 2012), we have introduced the so-called *Shapley II* value as follows.

Consider a scenario $\mathcal{S} = \emptyset, S_1, \dots, S_\tau$, and a particular transition $S_t \to S_{t+1}$ in $\mathcal{S}$. The players in the symmetric difference $S_t \Delta S_{t+1} = (S_t \setminus S_{t+1}) \cup (S_{t+1} \setminus S_t)$ are *active* at time t in $S_t \to S_{t+1}$ as they either leave or enter the current coalition S_t. The Shapley II value for player i is the sum of marginal contributions of i in each transition where i is active. We clarify this notion in more detail.

Suppose thus that $i \in S_t \Delta S_{t+1}$ is active. If i is the only active player at time t, the marginal contribution is simply $v(S_{t+1}) - v(S_t)$ because i's activity causes this change in v. If $|S_t \Delta S_{t+1}| \geq 2$, the Shapley II value decomposes $S_t \to S_{t+1}$ into *elementary* transitions (*i.e.*, transitions such that only one player enters or leaves) and considers all possibilities of doing so. For example, the transition $2 \to 13$, where player 2 leaves and players 1,3 enter, can be decomposed into $3! = 6$ different ways (so-called *paths*) corresponding to all permutations of players 1,2,3:

$$\begin{aligned}
&2 \to \emptyset \to 1 \to 13\\
&2 \to \emptyset \to 3 \to 13\\
&2 \to 12 \to 1 \to 13\\
&2 \to 12 \to 123 \to 13\\
&2 \to 23 \to 3 \to 13\\
&2 \to 23 \to 123 \to 13
\end{aligned}$$

In each path, the marginal contribution is computed as the difference $v(T') - v(T)$, where i is active in the elementary transition $T \to T'$. For example, the marginal contribution of player 1 is $v(1) - v(\emptyset)$ in the 1st path, $v(13) - v(3)$ in the second, while the marginal contribution of player 2 is $v(\emptyset) - v(2)$ for these two paths. Averaging on all paths, we obtain the following marginal contributions for transition $1 \to 23$:

$$\begin{aligned}
\phi_1^{1\to 23}(v) &= \frac{1}{6}v(1) + \frac{1}{3}(v(13) - v(3)) + \frac{1}{3}(v(12) - v(2)) + \frac{1}{6}(v(123) - v(23))\\
\phi_2^{1\to 23}(v) &= -\frac{1}{3}v(2) + \frac{1}{6}(v(1) - v(12)) + \frac{1}{3}(v(13) - v(123)) + \frac{1}{6}(v(3) - v(23))\\
\phi_3^{1\to 23}(v) &= \frac{1}{6}v(3) + \frac{1}{3}(v(13) - v(1)) + \frac{1}{6}(v(123) - v(12)) + \frac{1}{3}(v(23) - v(2)).
\end{aligned}$$

In summary, the Shapley II scenario-value is computed as follows:

$$\phi_i^{\mathcal{S}}(v) = \sum_{t | i \in S_t \Delta S_{t+1}} \phi_i^{S_t \to S_{t+1}}(v) \tag{1}$$

with

$$\phi_i^{S_t \to S_{t+1}}(v) = \frac{1}{|S_t \Delta S_{t+1}|!} \sum_{\mathcal{P} \text{ from } S_t \text{ to } S_{t+1}} (v(S'_{\mathcal{P}}) - v(S_{\mathcal{P}})) \tag{2}$$

where "$\mathcal{P}$ from S to T" is any path from S to T in 2^N, and $(S_{\mathcal{P}}, S'_{\mathcal{P}})$ is the unique edge (transition) of $\mathcal{P}$ such that either $\{i\} = S_{\mathcal{P}} \setminus S'_{\mathcal{P}}$ or $\{i\} = S'_{\mathcal{P}} \setminus S_{\mathcal{P}}$.

Notice that the computation of the marginal contribution in a transition $S \to T$ resembles the computation of the classical Shapley value relative to the set $S\Delta T$ of active players. We formalize this idea. Consider a transition $S \to T$. It is convenient to introduce the mapping

$$\Gamma_{S,T} : S\Delta T \to \widehat{ST}, \quad K \mapsto K\Delta S,$$

where $\widehat{ST}$ is the collection of sets in $S \cup T$ containing $S \cap T$ (observe that $K\Delta S$ always contains $S \cap T$). The inverse mapping $\Gamma^{-1} : \widehat{ST} \to S\Delta T$ is simply $K \mapsto K\Delta S$ again, and we have a bijection between $S\Delta T$ and $\widehat{ST}$.

Next we introduce the *local game* $v_{S,T}$ on the set of active players $S\Delta T$, defined by

$$v_{S,T}(K) = v(\Gamma(K)) - v(S) = v(K\Delta S) - v(S), \quad K \subseteq S\Delta T.$$

Also $v(K) = v_{S,T}(K\Delta S) + v(S)$ on $\widehat{ST}$.

Observe that if i is entering, then we have $\{i\} = S'_{\mathcal{P}} \setminus S_{\mathcal{P}}$ in the above notation. So the marginal contribution in $\phi_i^{S\to T}$ is $v(S_{\mathcal{P}} \cup i) - v(S_{\mathcal{P}})$. Since $i \notin S$, the marginal contribution is $v_{S,T}((S_{\mathcal{P}}\Delta S) \cup i) - v_{S,T}(S_{\mathcal{P}}\Delta S)$. If i is leaving, the marginal contribution in $\phi_i^{S\to T}$ is $v(S_{\mathcal{P}} \setminus i) - v(S_{\mathcal{P}})$. Since $i \in S$, however, the marginal contribution in terms of the local game is still $v_{S,T}((S_{\mathcal{P}}\Delta S) \cup i) - v_{S,T}(S_{\mathcal{P}}\Delta S)$, which is a term of the classical Shapley value of i in $v_{S,T}$, denoted by $\phi_i^{\mathrm{Sh}}(v_{S,T})$. Since Γ is a bijection, the computation of $\phi_i^{S\to T}(v)$ amounts to the computation of $\phi_i^{\mathrm{Sh}}(v_{S,T})$. We have shown:

$$\phi^{S\to T}(v) = \phi^{\mathrm{Sh}}(v_{S,T}). \tag{3}$$

where ϕ^{Sh} is the classical Shapley value.

3 Axiomatization of the Shapley II value

We denote by $\psi : \mathcal{G} \to \mathbb{R}^{n \times \mathfrak{S}}$ a scenario-value, where $\mathfrak{S}$ is the set of finite sequences of coalitions (not necessarily starting with $\emptyset$).

Two sequences $\mathcal{S} = S_1, \dots, S_q, \mathcal{S}' = S'_1, \dots, S'_r$ are said to be *concatenable* if $S_q = S'_1$, in which case their *concatenation* is the sequence

$$\mathcal{S} \oplus \mathcal{S}' := S_1, \dots, S_q, S'_2, \dots, S'_r.$$

Concatenation (C): Let $\mathcal{S}, \mathcal{S}'$ be two concatenable sequences. Then

$$\psi^{\mathcal{S}\oplus\mathcal{S}'} = \psi^{\mathcal{S}} + \psi^{\mathcal{S}'}.$$

Axiom (C) allows us to restrict our attention to transitions. Indeed,

$$\psi^{\mathcal{S}} = \sum_{k=0}^{t-1} \psi^{S_k \to S_{k+1}}$$

holds for every sequence $\mathcal{S} = S_0, S_1, \dots, S_t$.

Inactive players in transitions (IP): If i is inactive in $S \to T$, then $\psi_i^{S \to T}(v) = 0$ for any game v.

Efficiency for transitions (E): For any transition $S \to T$ and game v, we have

$$\sum_{i \in N} \psi_i^{S \to T}(v) = v(T) - v(S).$$

Linearity for transitions (L): $v \mapsto \psi^{S \to T}(v)$ is a linear operator for any transition $S \to T$.

Symmetry for transitions (S): For any $i \in N$, any transition $S \to T$ and any permutation σ on N, one has

$$\psi_i^{S \to T}(v) = \psi_{\sigma(i)}^{\sigma(S) \to \sigma(T)}(v \circ \sigma^{-1}).$$

We introduce the *signature* of a transition $S \to T$ as the parameter

$$\tau(S \to T) := (|S \setminus T|, |T \setminus S|, |S \cap T|).$$

As shown in Faigle and Grabisch (2012), the signature is invariant under permutations, and moreover, two scenarios are equal up to a permutation of the players if and only if they thave the same signature.

$i \in N$ is a *null player* for v if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$.

Null axiom for transitions (N): Every null player i obtains $\psi_i^{S \to T}(v) = 0$ relative to every transition $S \to T$.

Two players i, j are antisymmetric if $v(K \cup \{i, j\}) = v(K)$ for every $K \subseteq N \setminus \{i, j\}$.

Antisymmetry for entering/leaving players (ASEL): if $i \in S \setminus T$ and $j \in T \setminus S$ are antisymmetric for v, then $\psi_i^{S \to T}(v) = \psi_j^{S \to T}(v)$.

Antisymmetric players have in some sense a counterbalancing effect: they annihilate each other when entering together a coalition, which can be interpreted by saying that they bring the same contribution but of opposite sign. Therefore, if one is leaving and the other entering, their contribution in the scenario becomes equal and of same sign.

Theorem 1. A scenario-value satisfies (C), (L), (IP), (E), (S), (N) and (ASEL) if and only if it is the Shapley II scenario-value.

(see proof in Appendix)

An important point to note is that, in contrast to the classical case, two symmetry axioms are present. Relative to the transition $S \to T$, the first one, axiom (S), says that set of players can be freely permuted provided they all belong to one of the groups $S \setminus T$, $T \setminus S$, $S \cap T$, or $N \setminus (S \cup T)$. Now (IP) implies that we do not have to bother about players in $S \cap T$ and $N \setminus (S \cup T)$. The second symmetry axiom (ASEL) tells us how to exchange players between $S \setminus T$ and $T \setminus S$. Interestingly, however, both axioms can be deduced from the application of the classical symmetry axiom to the local game $v_{S,T}$. Indeed, consider two symmetric players $i, j \in S\Delta T$ for $v_{S,T}$, *i.e.*, $v_{S,T}(K \cup i) = v_{S,T}(K \cup j)$ holds for any $K \subseteq (S\Delta T) \setminus \{i, j\}$. In the case $i, j \in S \setminus T$, this yields

$$v((K\Delta S) \setminus i) = v((K\Delta S) \setminus j),$$

or, setting $K' = (K\Delta S) \setminus \{i, j\}$, $v(K' \cup i) = v(K' \cup j)$, which means symmetry of i, j for v for sets in $\widehat{ST} \setminus \{i, j\}$. If $i, j \in T \setminus S$, we have

$$v((K\Delta S) \cup i) = v((K\Delta S) \cup j)$$

which also exhibits symmetry of i, j for those sets. In rhe case $i \in S \setminus T$ and $j \in T \setminus S$, we obtain

$$v((K\Delta S) \setminus i) = v((K\Delta S) \cup j),$$

or, setting $K' = (K\Delta S) \setminus i$, $v(K') = v(K' \cup \{i, j\})$, for every $K' \subseteq \widehat{ST} \setminus \{i, j\}$. But this is precisely antisymmetry.

In the original paper, antisymmetric players were defined as players i, j satisfying

$$v(K \cup i) - v(K) = v(K \cup \{i, j\}) - v(K \cup i)$$
$$v(K \cup j) - v(K) = v(K \cup \{i, j\}) - v(K \cup j)$$

for any $K \subseteq N \setminus i, j$. Then two such antisymmetric players i, j satisfy $\psi_i^{S\to T}(v) = -\psi_j^{S\to T}(v)$ for any sequence $S \to T$ with $i \in S \setminus T$ and $j \in T \setminus S$. It can be checked that Shapley II does have this antisymmetric property. However, it is too weak to ensure uniqueness of the scenario value.

Another interesting property is the following one, which was not mentionned in Faigle and Grabisch (2012).

> **Changing Role (CR):** for any $S, T \subseteq N$, for any $i \in N \setminus (S \cup T)$, and any game v, we have $\psi_i^{S\cup i\to T}(v) = -\psi_i^{S\to T\cup i}(v)$.

Consider a transition $S \to T$ where player i is not participating (*i.e.*, $i \notin S \cup T$). Suppose now that player i joins T, that is, i becomes an entering player in the transition $S \to T \cup i$. Then i is active in this transition and has some marginal contribution, say α. In contrast, assume now that player i joins S and leaves during the transition, *i.e.*, we consider the transition $S \cup i \to T$. Then i is active in this transition and has some marginal contribution, say β. Note that $(S \cup i)\Delta T = S\Delta(T \cup i)$, which means that the set of active players is the same, only the rôle of i has been switched from entering to leaving, and the rest is left unchanged. Under these conditions, axiom (CR) says that the marginal contributions of i in these two transitions are opposite, *i.e.*, $\beta = -\alpha$.

We claim that Shapley II satisfies (CR). To establish the claim, we consider a transition $S \to T$ with $S \cup T \neq N$, and $i \in N \setminus (S \cup T)$. We know that for any v

$$\phi_i^{S\to T\cup i}(v) = \phi_i^{\mathrm{Sh}}(v_{S,T\cup i}), \quad \phi_i^{S\cup i\to T}(v) = \phi_i^{\mathrm{Sh}}(v_{S\cup i,T}).$$

Setting $\ell = |(S \cup i)\Delta T| = |S\Delta(T \cup i)$, $|K| = k$, we find:

$$\begin{aligned}
\phi_i^{\mathrm{Sh}}(v_{S\cup i,T}) &= \sum_{K\subseteq((S\cup i)\Delta T)\setminus i} \frac{(\ell-k-1)!k!}{\ell!}(v_{S\cup i,T}(K\cup i) - v_{S\cup i,T}(K)) \\
&= \sum_{K\subseteq S\Delta T} \frac{(\ell-k-1)!k!}{\ell!}(v(K\Delta S) - v((K\Delta S)\cup i)) \\
\phi_i^{\mathrm{Sh}}(v_{S,T\cup i}) &= \sum_{K\subseteq(S\Delta(T\cup i))\setminus i} \frac{(\ell-k-1)!k!}{\ell!}(v_{S,T\cup i}(K\cup i) - v_{S,T\cup i}(K)) \\
&= \sum_{K\subseteq S\Delta T} \frac{(\ell-k-1)!k!}{\ell!}(v((K\Delta S)\cup i) - v(K\Delta S)),
\end{aligned}$$

which proves the claim. However, it can be demonstrated that the axiomatization of Shapley II fails if (CR) replaces (ASEL).

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4 Proof of Theorem 1

($\Leftarrow$) We check that (ASEL) is satisfied by Shapley II, the rest is left to the reader. It suffices to establish the antisymmetry property for a transition $S \to T$. Let us argue that i, j being antisymmetric for v implies that i, j are symmetric in the classical sense for $v_{S,T}$, *i.e.*, $v_{S,T}(K \cup i) = v_{S,T}(K \cup j)$ holds for all K in $S\Delta T \setminus \{i, j\}$. Indeed, this would yield $\phi_i^{S\to T}(v) = \phi_i^{\mathrm{Sh}}(v_{S,T}) = \phi_j^{\mathrm{Sh}}(v_{S,T}) = \phi_j^{S\to T}(v)$, the desired result.

Now, $v(K \cup \{i, j\}) = v(K)$ for any $K \subseteq \widehat{ST} \setminus \{i, j\}$ is equivalent to

$$v_{S,T}((K\Delta S) \cup \{i, j\}) = v_{S,T}(K\Delta S)$$

or

$$v_{S,T}(K' \cup j) = v_{S,T}(K' \cup i)$$

with $K' = (K\Delta S) \setminus i$, which proves the claim.

($\Rightarrow$) Since (C) is satisfied, it suffices to derive an expression for transitions. Under (L), (S), (N), it is shown in (Faigle and Grabisch, 2012, Prop. 2) that the value takes the form:

$$\psi_i^{S\to T} = \begin{cases} \sum_{K\subseteq N\setminus i} a_{\tau(S\to T),\tau(S\to T|K\cup i)}(v(K\cup i) - v(K)), & \text{if } i \in S \setminus T \\ \sum_{K\subseteq N\setminus i} b_{\tau(S\to T),\tau(S\to T|K\cup i)}(v(K\cup i) - v(K)), & \text{if } i \in T \setminus S \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau(S \to T|K) := (|(S \setminus T) \cap K|, |(T \setminus S) \cap K|, |S \cap T \cap K|, |K \setminus (S \cup T)|)$, and $a_{\tau(S\to T),\tau(S\to T|K\cup i)}$ and $b_{\tau(S\to T),\tau(S\to T|K\cup i)}$ are real coefficients. Then axioms (IP) and (E) imply:

$$\begin{aligned}\sum_{i\in N} \psi_i^{S\to T}(v) &= v(T) - v(S) \\ &= \sum_{i\in S\setminus T} \sum_{K\subseteq N\setminus i} a_{\tau(S\to T),\tau(S\to T|K\cup i)}(v(K\cup i) - v(K)) \\ &+ \sum_{i\in T\setminus S} \sum_{K\subseteq N\setminus i} b_{\tau(S\to T),\tau(S\to T|K\cup i)}(v(K\cup i) - v(K)) \\ &= \sum_{K\subseteq N} v(K)\Big(k_l a_{\tau,k_l,k_r,k_c,k_0} + k_r b_{\tau,k_l,k_r,k_c,k_0} \\ &- (l - k_l)a_{\tau,k_l+1,k_r,k_c,k_0} - (r - k_r)b_{\tau,k_l,k_r+1,k_c,k_0}\Big),\end{aligned}$$

with the following notations: $\tau(S \to T) =: \tau$, $|S \setminus T| =: l$, $|T \setminus S| =: r$, $|S \cap T| =: c$, $|K| = k$, $\tau(S \to T|K) = (k_l, k_r, k_c, k_0)$, with $k_l = |(S \setminus T) \cap K|$, $k_r = |(T \setminus S) \cap K|$, $k_c = |S \cap T \cap K|$, and $k_0 = k - k_l - k_r - k_c = |K \setminus (S \cup T)|$. Let us drop also the subindex τ since it is present everywhere. This gives by identification:

$$la_{l,0,c,0} - rb_{l,1,c,0} = -1 \tag{4}$$

$$-la_{1,r,c,0} + rb_{0,r,c,0} = 1 \tag{5}$$

$$\begin{aligned}k_l a_{k_l,k_r,k_c,k_0} + k_r b_{k_l,k_r,k_c,k_0} &- (l - k_l)a_{k_l+1,k_r,k_c,k_0} \\ &-(r - k_r)b_{k_l,k_r+1,k_c,k_0} = 0, \quad \forall K \neq S, T.\end{aligned} \tag{6}$$

Note that $1 \leq k_l \leq l$ for a_{k_l,k_r,k_c,k_0}, $1 \leq k_r \leq r$ for $b_{k_l.k_r.k_c.k_0}$, and in (3) the configurations $(l_l, k_r, k_c, k_0) = (l, 0, c, 0)$ and $(0, r, c, 0)$ are excluded.

1. Suppose that $S \subset T$ holds, *i.e.*, $\tau = (0, t-s, s)$. Then $l = k_l = 0$, $r = t-s$, $c = s$, and (4), (5) yield $b_{0,1,s,0} = \frac{1}{t-s}$ and $b_{0,r,s,0} = \frac{1}{t-s}$, and the remaining equations become:

$$k_r b_{0,k_r,k_c,k_0} - (r-k_r) b_{0,k_r+1,k_c,k_0} = 0, \quad \forall K \neq S, T.$$

If $K \cap T \setminus S = \emptyset$, this reduces to

$$b_{0,1,k_c,k_0} = 0, \quad \forall k_c, k_0 \tag{7}$$

except the case $(k_c = s, k_0 = 0)$, which corresponding to S. Similarly, $K \supseteq T \setminus S$ yields

$$b_{0,t-s,k_c,k_0} = 0, \quad \forall k_c, k_0, \tag{8}$$

except in the case $(k_c = s, k_0 = 0)$, which corresponds to T.

So it remains to examine the case where all K satisfy $K \cap (T \setminus S) \neq \emptyset$ and $K \not\supseteq T \setminus S$ (*i.e.*, $0 < k_r < t-s$). We prove by induction that $b_{0,k_r+1,k_c,k_0} = 0$ holds for all $0 < k_r < t-s$ and k_c, k_0, except for $k_c = s, k_0 = 0$, *i.e.*, for $K = S \cup L$ with $\emptyset \neq L \subset T \setminus S$, where

$$b_{0,k_r+1,s,0} = \frac{k_r!}{(t-s)\cdots(t-s-k_r)}.$$

For $k_r = 1$, we have

$$b_{0,1,k_c,k_0} - (r-1) b_{0,2,k_c,k_0} = 0.$$

From (7) we get $b_{0,1,k_c,k_0} = 0$ except if $(k_c = s, k_0 = 0)$, which entails $b_{0,2,k_c,k_0} = 0$ for all k_c, k_0 except $b_{0,2,s,0} = \frac{1}{(t-s)(t-s-1)}$, the expected result. Assume that the assumption is true up to k_r and compute the case $k_r + 1$, assuming $k_r + 1 < t - s$. We find

$$(k_r+1) b_{0,k_r+1,k_c,k_0} - (r-k_r-1) b_{0,k_r+2,k_c,k_0} = 0$$

By the assumption, the first term vanishes for all k_c, k_0, except for $k_c = s$ and $k_0 = 0$. This implies the second term to vanish except when

$$b_{0,k_r+2,s,0} = \frac{(k_r+1)!}{(t-s)\cdots(t-s-k_r)(t-s-k_r-1)}.$$

Therefore, the expression of $\psi_i^{S\to T}$ becomes

$$\psi_i^{S\to T}(v) = \sum_{\substack{K \supseteq S \\ K \subseteq T \setminus i}} \frac{(t-s-k_r-1)!k_r!}{(t-s)!} (v(K \cup i) - v(K)),$$

which is the expression of the Shapley value for a game on the set $T \setminus S$.

2. The case $T \subset S$ is analyzed similarly.

3. It remains to settle the case where $S\setminus T \neq \emptyset$ and $T\setminus S \neq \emptyset$ hold. Take any $i \in S\setminus T$ and $j \in T\setminus S$ and suppose that there are antisymmetric for v, *i.e.*, $v(K\cup\{i,j\}) = v(K)$ for any $K \subseteq N\setminus\{i,j\}$. This yields

$$\begin{aligned}
\psi_i^{S\to T}(v) = & \sum_{K\subseteq N\setminus i} a_{k_l+1,k_r,k_c,k_0}(v(K\cup i) - v(K)) \\
= & \sum_{\substack{K\subseteq N\setminus i\\ K\ni j}} a_{k_l+1,k_r,k_c,k_0}(v(K\setminus j) - v(K)) + \sum_{\substack{K\subseteq N\setminus i\\ K\not\ni j}} a_{k_l+1,k_r,k_c,k_0}(v(K\cup i) - v(K)) \\
= & \sum_{K\subseteq N\setminus\{i,j\}} \Big(v(K)(a_{k_l+1,k_r+1,k_c,k_0} - a_{k_l+1,k_r,k_c,k_0}) + v(K\cup i)a_{k_l+1,k_r,k_c,k_0} \\
& + v(K\cup j)(-a_{k_l+1,k_r+1,k_c,k_o})\Big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\psi_j^{S\to T}(v) = & \sum_{\substack{K\subseteq N\setminus j\\ K\ni i}} b_{k_l,k_r+1,k_c,k_0}(v(K\setminus i) - v(K)) + \sum_{\substack{K\subseteq N\setminus j\\ K\not\ni i}} b_{k_l,k_r+1,k_c,k_0}(v(K\cup j) - v(K)) \\
= & \sum_{K\subseteq N\setminus\{i,j\}} \Big(v(K)(b_{k_l+1,k_r+1,k_c,k_0} - b_{k_l,k_r+1,k_c,k_0}) + v(K\cup i)(-b_{k_l+1,k_r+1,k_c,k_0}) + \\
& v(K\cup j)b_{k_l,k_r+1,k_c,k_o}\Big).
\end{aligned}$$

Since $\psi_i^{S\to T}(v) = \psi_j^{S\to T}(v)$ for any such game we deduce the system

$$\begin{aligned}
a_{k_l+1,k_r+1,k_c,k_0} - a_{k_l+1,k_r,k_c,k_0} &= b_{k_l+1,k_r+1,k_c,k_0} - b_{k_l,k_r+1,k_c,k_0} \\
a_{k_l+1,k_r,k_c,k_0} &= -b_{k_l+1,k_r+1,k_c,k_0} \\
a_{k_l+1,k_r+1,k_c,k_0} &= -b_{k_l,k_r+1,k_c,k_0},
\end{aligned}$$

for $0 \le k_l \le l-1$, $0 \le k_r \le r-1$, $0 \le k_c \le c$, and $0 \le k_0 \le n - |S\cup T|$, with the above conventions. Remark that the first line is redundant. Substituting in (4), (5) we obtain

$$a_{l,0,c,0} = -\frac{1}{l+r}, \quad a_{1,r,c,0} = -\frac{1}{l+r}. \tag{9}$$

Substitution into (6) leads to

$$(k_l + r - k_r)a_{k_l,k_r,k_c,k_0} - (k_r + l - k_l)a_{k_l+1,k_r,k_c,k_0} = 0 \tag{10}$$

with the restriction $1 \le k_l \le l-1$, $1 \le k_r \le r-1$. For the remaining cases, we get:

$$\begin{aligned}
-(l+k_r)a_{1,k_r,k_c,k_0} + (r-k_r)a_{1,k_r+1,k_c,k_0} &= 0, \quad k_l = 0, \quad 0 \le k_r \le r & (11)\\
(k_l+r)a_{k_l,0,k_c,k_0} - (l-k_l)a_{k_l+1,0,k_c,k_0} &= 0, \quad 1 \le k_l \le l-1, \quad k_r = 0 & (12)\\
k_l a_{k_l,r,k_c,k_0} - (l-k_l+r)a_{k_l+1,r,k_c,k_0} &= 0, \quad 1 \le k_l \le l-1, \quad k_r = r & (13)\\
(l+r-k_r)a_{l,k_r,k_c,k_0} - k_r a_{l,k_r-1,k_c,k_0} &= 0, \quad k_l = l, \quad 0 \le k_r \le r, & (14)
\end{aligned}$$

where in (11) the case $(k_r = r, k_c = c, k_0 = 0)$ is excluded, and in (14) the case $(k_r = 0, k_c = c, k_0 = 0)$ is excluded.

We claim that all coefficients corresponding to $K \setminus (S \cup T) \neq \emptyset$ (*i.e.*, $k_0 > 0$) or $K \not\supseteq (S \cap T)$ (*i.e.*, $k_c < c$) vanish. Suppose then that $k_0 > 0$ and $k_c < c$ is given. From (14) with $k_r = 0$, we deduce $a_{l,0,k_c,k_0} = 0$. Substitution in (12) with $k_l = l - 1$ yields $a_{l-1,0,k_c,k_0} = 0$. Successive application of (12), again with $k_l = l - 2, \dots, 1$, yields

$$a_{k_l,0,k_c,k_0} = 0, \quad 1 \leq k_l \leq l.$$

Since $a_{l,0,k_c,k_0}$ is also present in (14) with $k_r = 1$, we have $a_{l,1,k_c,k_0} = 0$. Now, in (10), $a_{l,1,k_c,k_0}$ is present with $(k_l = l - 1, k_r = 1)$, which yields $a_{l-1,1,k_c,k_0} = 0$. Applying again (10) with $k_l = l - 2, \dots, 1$ we deduce

$$a_{k_l,1,k_c,k_0} = 0, \quad 1 \leq k_l \leq l.$$

$a_{l,1,k_c,k_0}$ is present also in (14) with $k_r = 2$. Proceeding as above we get

$$a_{k_l,2,k_c,k_0} = 0, \quad 1 \leq k_l \leq l.$$

This can be done until $k_r = r$ in (14), which gives $a_{l,r,k_c,k_0} = 0$. Then (13) has to be used with $k_l = l - 1$ and so on. This yields

$$a_{k_l,r,k_c,k_0} = 0, \quad 1 \leq k_l \leq l.$$

In summary, $a_{k_l,k_r,k_c,k_0} = 0$ holds for $1 \leq k_l \leq l$, $0 \leq k_r \leq r$, $0 \leq k_c < c$ and $0 < k_0 \leq n - |S \cup T|$, and our claim is proved.

Substituting into (10) to (14), we find

$$\begin{aligned}
(k_l + r - k_r)a_{k_l,k_r,c,0} - (k_r + l - k_l)a_{k_l+1,k_r,c,0} &= 0, \quad 1 \leq k_l \leq l-1, 1 \leq k_r \leq r-1 && (15)\\
-(l + k_r)a_{1,k_r,c,0} + (r - k_r)a_{1,k_r+1,c,0} &= 0, \quad 0 \leq k_r \leq r-1 && (16)\\
(k_l + r)a_{k_l,0,c,0} - (l - k_l)a_{k_l+1,0,c,0} &= 0, \quad 1 \leq k_l \leq l-1 && (17)\\
k_l a_{k_l,r,c,0} - (l - k_l + r)a_{k_l+1,r,c,0} &= 0, \quad 1 \leq k_l \leq l-1 && (18)\\
(l + r - k_r)a_{l,k_r,c,0} - k_r a_{l,k_r-1,c,0} &= 0, \quad 1 \leq k_r \leq r. && (19)
\end{aligned}$$

Observe that the system (16) together with $a_{1,r,c,0} = -\frac{1}{l+r}$ is a triangular system of $r+1$ equations in $r+1$ variables $a_{1,0,c,0}, \dots, a_{1,r,c,0}$. It has therefore a unique solution. The same observation applies to the systems (17) with $a_{l,0,c,0} = -\frac{1}{l+r}$, (18) with $a_{1,r,c,0} = -\frac{1}{l+r}$, and (19) with $a_{l,0,c,0} = -\frac{1}{l+r}$, which determines in a unique way the variables $a_{1,0,c,0}, \dots, a_{l,0,c,0}$, $a_{1,r,c,0}, \dots, a_{l,r,c,0}$, and $a_{l,0,c,0}, \dots, a_{l,r,c,0}$ respectively.

Substituting into the system (15), we find a system of $(l-1)(r-1)$ equations in the $(l-1)(r-1)$ variables $a_{2,1,c,0}, \dots, a_{2,r-1,c,0}, a_{3,1,c,0}, \dots, a_{3,r-1,c,0}, \dots, a_{l-1,1,c,0}, \dots, a_{l-1,r-1,c,0}$, which is triangular and consequently has a unique solution. Since we know that the coefficients of the Shapley II scenario value satisfy (15) to (19), it is the unique solution.